Note on Lines of Curvature.

By Thomas Hardy Taliaferro.

In a note in the Comptes Rendus for March 25th, 1895, Professor Craig has given a condition for the determination of surfaces having lines of curvature corresponding to a system of conjugate lines on a given surface.

Suppose the surface to be represented by the equations

$$\begin{cases} x = f_1(\rho, \rho_1), \\ y = f_2(\rho, \rho_1), \\ z = f_3(\rho, \rho_1), \end{cases}$$

where ρ, ρ_1 are the parameters of a system of conjugate lines; then in the note referred to it is shown how surfaces can be found whose coordinates are given by

$$X = \psi_1(x), \quad Y = \psi_2(y), \quad Z = \psi_3(z),$$

on which the original conjugate lines are lines of curvature.

The condition to be satisfied is of course

$$\frac{\partial X}{\partial \rho}\frac{\partial X}{\partial \rho_1} + \frac{\partial Y}{\partial \rho}\frac{\partial Y}{\partial \rho_1} + \frac{\partial Z}{\partial \rho}\frac{\partial Z}{\partial \rho_1} = 0. \tag{1}$$

The first difficulty in the problem is in finding an initial surface whose coordinates are given explicitly as functions of the parameters of a system of conjugate lines. Certain methods are known for this, especially the elegant method of Koenigs (Darboux, Vol. I, page 112), but all are very difficult of application in any particular case.

I have ventured in the following brief note to give a simple application of the problem to the case of tetrahedral surfaces where $m = n$, and also to give two examples. The tetrahedral surfaces are given by the equations (Darboux, Vol. I, page 142):

$$\left.\begin{aligned} x &= \lambda A(\rho - a)^m(\rho_1 - a)^n, \\ y &= \mu B(\rho - b)^m(\rho_1 - b)^n, \\ z &= \nu C(\rho - c)^m(\rho_1 - c)^n, \end{aligned}\right\} \tag{2}$$

where ρ, ρ_1 are the parameters of a system of conjugate lines; m, n, A, B, C any real constants; and λ, μ, ν either 1 or i.

The left-hand members of (2) all satisfy the equation

$$(\rho_1 - \rho)\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1} + n\frac{\partial\theta}{\partial\rho} - m\frac{\partial\theta}{\partial\rho_1} = 0, \tag{3}$$

since ρ, ρ_1 are the parameters of a system of conjugate lines, but as $x^2 + y^2 + z^2$ does not satisfy equation (3), ρ, ρ_1 are not the parameters of the lines of curvature. It is readily seen that the condition to be satisfied in order that ρ, ρ_1 should be the parameters of the lines of curvature is

$$\begin{aligned}\lambda^2A^2(\rho-a)^{2m-1}(\rho_1-a)^{2n-1} + \mu^2B^2(\rho-b)^{2m-1}(\rho_1-b)^{2n-1}&\\ + \nu^2C^2(\rho-c)^{2m-1}(\rho_1-c)^{2n-1} &= 0.\end{aligned} \tag{4}$$

I.—*Tetrahedral Surfaces, when* $m = n$.

When $m = n$, the expressions for the cartesian coordinates x, y, z of the tetrahedral surfaces in terms of the parameters ρ, ρ_1 of a system of conjugate lines become

$$\left.\begin{aligned} x &= \lambda A(\rho-a)^m(\rho_1-a)^m,\\ y &= \mu B(\rho-b)^m(\rho_1-b)^m,\\ z &= \nu C(\rho-c)^m(\rho_1-c)^m.\end{aligned}\right\} \tag{5}$$

The equation of condition (4) becomes

$$\begin{aligned}\lambda^2A^2(\rho-a)^{2m-1}(\rho_1-a)^{2m-1} + \mu^2B^2(\rho-b)^{2m-1}(\rho_1-b)^{2m-1}&\\ + \nu^2C^2(\rho-c)^{2m-1}(\rho_1-c)^{2m-1} &= 0.\end{aligned} \tag{6}$$

The equation of the tetrahedral surface on eliminating ρ, ρ_1 is readily seen to be of the form

$$\alpha\left(\frac{x}{\lambda A}\right)^{1/m} - \beta\left(\frac{y}{\mu B}\right)^{1/m} + \gamma\left(\frac{z}{\nu C}\right)^{1/m} = 1, \tag{7}$$

where

$$\left\{\begin{aligned}\alpha &= \frac{1}{(a-b)(a-c)},\\ \beta &= \frac{1}{(a-b)(b-c)},\\ \gamma &= \frac{1}{(b-c)(a-c)},\end{aligned}\right.$$

$$\alpha - \beta + \gamma = 0.$$

On adopting the convention

$$a > b > c,$$

it is seen that α, β, γ are real, positive quantities fulfilling the condition

$$\alpha < \beta < \gamma.$$

I say that writing

$$X = k_1 x^{1/2m}, \quad Y = k_2 y^{1/2m}, \quad Z = \Phi(z),$$

where k_1, k_2 are arbitrary constants, $\Phi(z)$ can be so determined by means of Craig's formula that on the derived surface ρ, ρ_1 will be the parameters of the lines of curvature, and furthermore that the derived surface will be a quadric surface depending on A, B, C, λ, μ, ν, k_1, k_2 for its form.

Since equation (1) consists of a single equation between three quantities, two of them may be assumed and the third determined.

Let

$$\psi_1(x) = k_1 x^{1/2m}, \quad \psi_2(y) = k_2 y^{1/2m}, \quad \psi_3(z) = \Phi(z), \tag{8}$$

where $\Phi(z)$ is to be determined.

Substituting these values (8) in equation (1), the following equation is derived:

$$\frac{1}{4}(\lambda A)^{1/m} k_1^2 + \frac{1}{4}(\mu B)^{1/m} k_2^2 + m^2\nu^2 C^2 (\rho - c)^{2m-1}(\rho_1 - c)^{2m-1}\left(\frac{d\Phi}{dz}\right)^2 = 0,$$

$$d\Phi = \frac{\pm i}{2m\nu C}\{(\lambda A)^{1/m} k_1^2 + (\mu B)^{1/m} k_2^2\}^{\frac{1}{2}} \frac{dz}{\{(\rho - c)^{2m-1}(\rho_1 - c)^{2m-1}\}^{\frac{1}{2}}},$$

$$dz = \frac{\partial z}{\partial \rho} d\rho + \frac{\partial z}{\partial \rho_1} d\rho_1,$$

$$\therefore \quad d\Phi = \pm i\{(\lambda A)^{1/m} k_1^2 + (\mu B)^{1/m} k_2^2\}^{\frac{1}{2}} d\{(\rho - c)^{\frac{1}{2}}(\rho_1 - c)^{\frac{1}{2}}\},$$

$$\therefore \ \Phi(z) = \frac{\pm i z^{1/2m}}{(\mu C)^{1/2m}}\{(\lambda A)^{1/m} k_1^2 + (\mu B)^{1/m} k_2^2\}^{\frac{1}{2}}. \tag{9}$$

The derived surface has for its cartesian coordinates X, Y, Z the following expressions:

$$\left.\begin{aligned} X &= k_1(\lambda A)^{1/2m}(\rho - a)^{\frac{1}{2}}(\rho_1 - a)^{\frac{1}{2}}, \\ Y &= k_2(\mu B)^{1/2m}(\rho - b)^{\frac{1}{2}}(\rho_1 - b)^{\frac{1}{2}}, \\ Z &= \pm i\{k_1^2(\lambda A)^{1/m} + k_2^2(\mu B)^{1/m}\}^{\frac{1}{2}}(\rho - c)^{\frac{1}{2}}(\rho_1 - c)^{\frac{1}{2}}. \end{aligned}\right\} \tag{10}$$

Equations (1) and (3) are satisfied, and the equation of the derived surface on which ρ, ρ_1 are the parameters of the lines of curvature is

$$\alpha \frac{X^2}{k_1^2(\lambda A)^{1/m}} - \beta \frac{Y^2}{k_2^2(\mu B)^{1/m}} - \gamma \frac{Z^2}{k_1^2(\lambda A)^{1/m} + k_2^2(\mu B)^{1/m}} = 1. \tag{11}$$

Equation (11) is the equation of a quadric surface depending on A, B, C, λ, μ, ν, k_1, k_2 for its form.

II.—*Examples.*

1. $m = n = \frac{1}{2}$, $\lambda = \nu = 1$, $\mu = i$.

Equations (5) become

$$\begin{cases} x = A(\rho - a)^{\frac{1}{2}}(\rho_1 - a)^{\frac{1}{2}}, \\ y = iB(\rho - b)^{\frac{1}{2}}(\rho_1 - b)^{\frac{1}{2}}, \\ z = C(\rho - c)^{\frac{1}{2}}(\rho_1 - c)^{\frac{1}{2}}. \end{cases}$$

Equation (7) becomes

$$\alpha \frac{X^2}{A^2} + \beta \frac{y^2}{B^2} + \gamma \frac{z^2}{C^2} = 1,$$

which is the equation of an ellipsoid.

The equation of condition (6) that ρ, ρ_1 should be lines of curvature on the original surface reduces in the case of the ellipsoid to

$$A^2 - B^2 + C^2 = 0.$$

Equation (8) becomes

$$\psi_1(x) = k_1 x, \quad \psi_2(y) = k_2 y, \quad \psi_3(z) = \Phi(z),$$
$$\Phi(z) = \pm \frac{(k_2^2 B^2 - k_1^2 A^2)^{\frac{1}{2}} z}{C}.$$

The derived surface has for its cartesian coordinates X, Y, Z the following expressions:

$$\begin{cases} X = k_1 A(\rho - a)^{\frac{1}{2}}(\rho_1 - a)^{\frac{1}{2}}, \\ Y = ik_2 B(\rho - b)^{\frac{1}{2}}(\rho_1 - b)^{\frac{1}{2}}, \\ Z = \pm(k_2^2 B^2 - k_1^2 A^2)^{\frac{1}{2}}(\rho - c)^{\frac{1}{2}}(\rho_1 - c)^{\frac{1}{2}}. \end{cases}$$

The equation of the derived surface becomes

$$\alpha \frac{X^2}{k_1^2 A^2} + \beta \frac{Y^2}{k_2^2 B^2} + \gamma \frac{Z^2}{k_2^2 B^2 - k_1^2 A^2} = 1,$$

which is a quadric surface.

Making certain suppositions on k_1, k_2 the following surfaces are derived:

$$k_1 = 1,\ k_2 = 1, \qquad \alpha\frac{X^2}{A^2} + \beta\frac{Y^2}{B^2} + \gamma\frac{Z^2}{B^2 - A^2} = 1,\ B > A, \text{ an ellipsoid;}$$

$$k_1 = i,\ k_2 = 1, \quad -\alpha\frac{X^2}{A^2} + \beta\frac{Y^2}{B^2} + \gamma\frac{Z^2}{B^2 + A^2} = 1, \text{ an hyperboloid of one sheet;}$$

$$k_1 = 1,\ k_2 = i, \qquad \alpha\frac{X^2}{A^2} - \beta\frac{Y^2}{B^2} - \gamma\frac{Z^2}{B^2 + A^2} = 1, \text{ an hyperboloid of two sheets.}$$

Writing the equation of condition (1) for the case of the ellipsoid, it becomes

$$\left(\frac{d\psi_1}{dx}\right)^2 A^2 - \left(\frac{d\psi_2}{dy}\right)^2 B^2 + \left(\frac{d\psi_3}{dz}\right)^2 C^2 = 0. \tag{12}$$

On writing

$$\frac{d\psi_1}{dx} = BC, \quad \frac{d\psi_2}{dy} = \sqrt{2}\,CA, \quad \frac{d\psi_3}{dz} = AB,$$

equation (12) is satisfied and the values of ψ_1, ψ_2, ψ_3 are

$$\psi_1(x) = BCx, \quad \psi_2(y) = \sqrt{2}\,CAy, \quad \psi_3(z) = ABz. \tag{13}$$

The expressions for the cartesian coordinates X, Y, Z of the derived surface on which ρ, ρ_1 are the parameters of the lines of curvature are

$$\left.\begin{aligned} X &= D(\rho - a)^{\frac{1}{2}}(\rho_1 - a)^{\frac{1}{2}}, \\ Y &= \sqrt{2}\,iD(\rho - b)^{\frac{1}{2}}(\rho_1 - b)^{\frac{1}{2}}, \\ Z &= D(\rho - c)^{\frac{1}{2}}(\rho_1 - b)^{\frac{1}{2}}, \end{aligned}\right\} \tag{14}$$
$$D = ABC.$$

The equation of the derived surface is represented by

$$\alpha\frac{X^2}{D^2} + \beta\frac{Y^2}{2D^2} + \gamma\frac{Z^2}{D^2} = 1, \tag{15}$$

which is an ellipsoid.

On writing $\lambda = 1$, $\mu = \nu = i$, the initial equation becomes an hyperboloid of one sheet, and for $\lambda = \mu = 1$, $\nu = i$, the initial equation is that of an hyperboloid of two sheets. In both cases there can be derived, as in case of ellipsoid, quadric surfaces depending on k_1, k_2, A, B, C for their form.

2. $m = n = 2, \ \lambda = \mu = \nu = 1$.

Equations (5) become

$$\begin{cases} x = A\,(\rho - a)^2\,(\rho_1 - a)^2, \\ y = B\,(\rho - b)^2\,(\rho_1 - b)^2, \\ z = C\,(\rho - c)^2\,(\rho_1 - c)^2. \end{cases}$$

Equation (7) becomes

$$\alpha\left(\frac{x}{A}\right)^{\frac{1}{2}} - \beta\left(\frac{y}{B}\right)^{\frac{1}{2}} + \gamma\left(\frac{z}{C}\right)^{\frac{1}{2}} = 1,$$

which is the equation of Steiner's surface.

The equation of condition (6) that ρ, ρ_1 should be the parameters of the lines of curvature on the original surface reduces in the case of Steiner's surface to

$$A^2\,(\rho - a)^3\,(\rho_1 - a)^3 + B^2\,(\rho - b)^3\,(\rho_1 - b)^3 + C^2\,(\rho - c)^3\,(\rho_1 - c)^3 = 0.$$

Equation (8) becomes

$$\psi_1(x) = k_1 x^{\frac{1}{4}}, \quad \psi_2(y) = k_2 y^{\frac{1}{4}}, \quad \psi_3(z) = \Phi(z),$$

$$\Phi(z) = \pm\frac{iz^{\frac{1}{4}}}{C^{\frac{1}{4}}}\sqrt{\sqrt{A} + \sqrt{B}}.$$

The derived surface has for its cartesian coordinates X, Y, Z the following expressions:

$$\begin{cases} X = k_1\,A^{\frac{1}{4}}\,(\rho - a)^{\frac{1}{2}}\,(\rho_1 - a)^{\frac{1}{2}}, \\ Y = k_2\,B^{\frac{1}{4}}\,(\rho - b)^{\frac{1}{2}}\,(\rho_1 - b)^{\frac{1}{2}}, \\ Z = \pm\, i\,\{k_1^2\,A^{\frac{1}{2}} + k_2^2\,B^{\frac{1}{2}}\}^{\frac{1}{2}}\,(\rho - c)^{\frac{1}{2}}\,(\rho_1 - c)^{\frac{1}{2}}. \end{cases}$$

The equation of the derived surface becomes

$$\alpha\frac{X^2}{k_1^2\,A^{\frac{1}{2}}} - \beta\frac{Y^2}{k_2\,B^{\frac{1}{2}}} - \gamma\frac{Z^2}{k_1^2\,A^{\frac{1}{2}} + k_2^2\,B^{\frac{1}{2}}} = 1,$$

which is a quadric surface.

Making certain suppositions on k_1, k_2, the following surfaces are derived:

$k_1 = 1$, $k_2 = i$, $\quad \alpha\dfrac{X^2}{A^{\frac{1}{2}}} + \beta\dfrac{Y^2}{B^{\frac{1}{2}}} + \gamma\dfrac{Z^2}{B^{\frac{1}{2}} - A^{\frac{1}{2}}} = 1$, $B > A$ an ellipsoid;

$k_1 = i$, $k_2 = i$, $\quad -\alpha\dfrac{X^2}{A^{\frac{1}{2}}} + \beta\dfrac{Y^2}{B^{\frac{1}{2}}} + \gamma\dfrac{Z^2}{B^{\frac{1}{2}} + A^{\frac{1}{2}}} = 1$, an hyperboloid of one sheet;

$k_1 = 1$, $k_2 = 1$, $\quad \alpha\dfrac{X^2}{A^{\frac{1}{2}}} - \beta\dfrac{Y^2}{B^{\frac{1}{2}}} - \gamma\dfrac{Z^2}{B^{\frac{1}{2}} + A^{\frac{1}{2}}} = 1$, an hyperboloid of two sheets.

Writing the equation of condition (1) for the case of Steiner's surface, it becomes

$$A^2 = \left(\frac{d\psi_1}{dx}\right)^2 (\rho - a)^3 (\rho_1 - a)^3 + B^2 \left(\frac{d\psi_2}{dy}\right)^2 (\rho - b)^3 (\rho_1 - b)^3 \\ + C^2 \left(\frac{d\psi_3}{dz}\right)^2 (\rho - c)^3 (\rho_1 - c)^3 = 0. \quad (16)$$

On writing

$$\frac{d\psi_1}{dx} = A^{\frac{3}{4}} B C x^{-\frac{3}{4}}, \quad \frac{d\psi_2}{dy} = \sqrt{2}\, i A B^{\frac{3}{4}} C y^{-\frac{3}{4}}, \quad \frac{d\psi_3}{dz} = A B C^{\frac{3}{4}} z^{-\frac{3}{4}},$$

equation (16) is satisfied and the values of ψ_1, ψ_2, ψ_3 are

$$\psi_1(x) = 4A^{\frac{3}{4}} B C x^{\frac{1}{4}}, \quad \psi_2(y) = 4\sqrt{2}\, i A B^{\frac{3}{4}} C y^{\frac{1}{4}}, \quad \psi_3(z) = 4 A B C^{\frac{3}{4}} z^{\frac{1}{4}}. \quad (17)$$

The expressions for the cartesian coordinates X, Y, Z of the derived surface on which ρ, ρ_1 are the parameters of the lines of curvature are

$$\left.\begin{aligned} X &= D (\rho - a)^{\frac{1}{2}} (\rho_1 - a)^{\frac{1}{2}}, \\ Y &= \sqrt{2}\, i D (\rho - b)^{\frac{1}{2}} (\rho_1 - b)^{\frac{1}{2}}, \\ Z &= D (\rho - c)^{\frac{1}{2}} (\rho_1 - c)^{\frac{1}{2}}, \end{aligned}\right\} \quad (18)$$

$$D = 4ABC.$$

The equation of the derived surface is represented by

$$\alpha \frac{X^2}{D^2} + \beta \frac{Y^2}{2D^2} + \gamma \frac{Z^2}{D^2} = 1, \quad (19)$$

which is an ellipsoid.

In the case of quadric surfaces, and also of Steiner's surface, there can be derived other quadric surfaces by assuming any other two of the ψ's and determining the remaining one as above.

It is also readily seen that it is necessary and sufficient for X, Y, Z to have the following values:

$$X = k_1 x^{1/2m}, \quad Y = k_2 y^{1/2m}, \quad Z = \Phi(z),$$

in order that $d\Phi(z)$ should be an exact differential in the case of tetrahedral surfaces when $m = n$.

For write $$X = k_1 x^{t/m}, \quad Y = k_2 y^{t/m}, \quad Z = \Phi(z), \quad (20)$$

where t is any constant, then the following expression for $d\Phi(z)$ is derived from equation (1):

$$d\Phi(z) = \pm it\{k_1^2(\lambda A)^{2t/m}(\rho-a)^{2t-1}(\rho_1-a)^{2t-1} + k_2^2(\mu B)^{2t/m}(\rho-b)^{2t-1}(\rho_1-b)^{2t-1}\}^{\frac{1}{2}}\left\{\left(\frac{\rho_1-c}{\rho-c}\right)^{\frac{1}{2}}d\rho + \left(\frac{\rho-c}{\rho_1-c}\right)^{\frac{1}{2}}d\rho_1\right\}. \quad (21)$$

Write

$$it\{k_1^2(\lambda A)^{2t/m}(\rho-a)^{2t-1}(\rho_1-a)^{2t-1} + k_2^2(\mu B)^{2t/m}(\rho-b)^{2t-1}(\rho_1-b)^{2t-1}\}^{\frac{1}{2}} = U, \quad (22)$$

the condition for the integrability of equation (21) is

$$\frac{\partial}{\partial\rho_1}U\left(\frac{\rho_1-c}{\rho-c}\right)^{\frac{1}{2}} - \frac{\partial}{\partial\rho}U\left(\frac{\rho-c}{\rho_1-c}\right)^{\frac{1}{2}} = 0, \quad (23)$$

or

$$\left(\frac{\rho_1-c}{\rho-c}\right)^{\frac{1}{2}}\frac{\partial U}{\partial\rho_1} - \left(\frac{\rho-c}{\rho_1-c}\right)^{\frac{1}{2}}\frac{\partial U}{\partial\rho} = 0.$$

Since in general

$$\frac{\rho_1-c}{\rho-c} \neq \frac{\rho-c}{\rho_1-c},$$

it is necessary, in order to satisfy equation (23), that

$$\frac{\partial U}{\partial\rho} = 0, \quad \frac{\partial U}{\partial\rho_1} = 0,$$

or

$$\left.\begin{aligned}\frac{\partial}{\partial\rho}it\{k_1^2(\lambda A)^{2t/m}(\rho-a)^{2t-1}(\rho_1-a)^{2t-1}+k_2^2(\mu B)^{2t/m}(\rho-b)^{2t-1}(\rho_1-b)^{2t-1}\}^{\frac{1}{2}} = 0,\\ \frac{\partial}{\partial\rho_1}it\{k_1^2(\lambda A)^{2t/m}(\rho-a)^{2t-1}(\rho_1-a)^{2t-1}+k_2^2(\mu B)^{2t/m}(\rho-b)^{2t-1}(\rho_1-b)^{2t-1}\}^{\frac{1}{2}} = 0,\end{aligned}\right\} (24)$$

which requires

$$it\{k_1^2(\lambda A)^{2t/m}(\rho-a)^{2t-1}(\rho_1-a)^{2t-1}+k_2^2(\mu B)^{2t/m}(\rho-b)^{2t-1}(\rho_1-b)^{2t-1}\}^{\frac{1}{2}} = \text{const.}, \quad (25)$$

for which it is necessary and sufficient that $t = \frac{1}{2}$. Hence

$$X = k_1 x^{1/2m}, \quad Y = k_2 y^{1/2m}, \quad Z = \Phi(z).$$

Other special cases may arise where $t \neq \frac{1}{2}$, but they will obviously require some relation to exist between the constants in equation (25).

JOHNS HOPKINS UNIVERSITY, *May* 1*st*, 1895.